MATH1010 Assignment 6

Suggested Solution

1. (a) $F'(x) = \dfrac{\cos x}{x}$

   (b) $F'(x) = -\sqrt{1+x^2}$

   (c) $F'(x) = 3x^2 e^{x^6}$

   (d) $F'(x) = 2(\ln 2x)^2 - (lnx)^2$

   (e) $F'(x) = 3x^2 e^{\cos x^3} - 2xe^{\cos x^2}$

   (f) $$\begin{aligned} F'(x) &= 2 \cdot \frac{\sin\sqrt{\ln x}}{\sqrt{\ln x}} \cdot \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} \\ &= \frac{\sin\sqrt{\ln x}}{x\ln x} \end{aligned}$$

2. (a)

$$\begin{aligned} &\lim_{h\to 0} \frac{1}{h}\int_0^{\sin h} \sin(\sqrt{t^2+t^4})\,dt \\ =&\lim_{h\to 0} \frac{\sin(\sqrt{\sin^2 h + \sin^4 h})\cos h}{1} \\ =&\ 0 \end{aligned}$$

   (b)

$$\begin{aligned} &\lim_{h\to 0} \frac{1}{h\sin h}\int_0^{h^2} e^{t^2}\,dt \\ =&\lim_{h\to 0} \frac{2he^{h^4}}{\sin h + h\cos h} \\ =&\lim_{h\to 0} \frac{2e^{h^4} + 2he^{h^4}(4h^3)}{\cos h + \cos h - h\sin h} \\ =&1 \end{aligned}$$

(c)

$$\begin{aligned}
&\lim_{h\to 0}\frac{1}{h}\int_{-h}^{h}|\sqrt[3]{\sin^5(t)}|\,dt\\
=&\lim_{h\to 0}\frac{2|\sqrt[3]{\sin^5(t)}|}{1}\\
=&0
\end{aligned}$$

(d)

$$\begin{aligned}
&\lim_{h\to 0^+}\frac{1}{\ln(1+h)}\int_{2}^{3h+2}\sqrt{t^6+2t^4+3t^2+4}\,dt\\
=&\lim_{h\to 0^+}\frac{3\sqrt{(3h+2)^6+2(3h+2)^4+3(3h+2)^2+4}}{\frac{1}{1+h}}\\
=&\frac{3\sqrt{2^6+2^5+3\cdot 2^2+4}}{1}\\
=&12\sqrt{7}
\end{aligned}$$

3. (a)

$$\begin{aligned}
&\int_{x^{-1}}^{x}\cos(\sqrt{xt})\,dt &&\text{Put}\quad xt=u\\
=&\int_{1}^{x^2}\cos(\sqrt{u})\frac{1}{x}\,du &&xdt=du\\
=&\frac{1}{x}\int_{1}^{x^2}\cos(\sqrt{u})\,du
\end{aligned}$$

(b)

$$\begin{aligned}
&f'(x)=-\frac{1}{x^2}\int_{1}^{x^2}\cos(\sqrt{u})\,du+\frac{1}{x}\cos(\sqrt{x^2})(2x)\\
&f'(1)=2\cos(1)
\end{aligned}$$

4. (a)

$$f(x) = \int_{-2}^{x^3} x\sqrt{t^4+t+1}\,dt$$
$$= x\int_{-2}^{x^3} \sqrt{t^4+t+1}\,dt$$
$$f'(x) = \int_{-2}^{x^3} \sqrt{t^4+t+1}\,dt + x(3x^2)\sqrt{x^{12}+x^3+1}$$
$$\int_{-2}^{x^3} \sqrt{t^4+t+1}\,dt + 3x^3\sqrt{x^{12}+x^3+1}$$

(b) $f('x) = 2|\cos(x)|^{\frac{7}{2}}$

(c)

$$\int_{x}^{x^2} \sin(\frac{t}{x^2})\,dt \qquad \text{Put} \quad u = \frac{t}{x^2}$$
$$= x^2\int_{\frac{1}{x}}^{1} \sin(u)\,du \qquad du = \frac{1}{x^2}dt$$
$$f'(x) = 2x\int_{\frac{1}{x}}^{1} \sin(u)\,du - x^2(-\frac{1}{x^2})\sin(\frac{1}{x})$$
$$= -2x\cos(1) + 2x\cos(\frac{1}{x}) + \sin(\frac{1}{x})$$

(d)

$$f'(x) = \int_{10}^{x} \frac{dt}{1+t^4+\sin^4 t}$$

5. Since $\int_0^x f(t)\,dt = \int_x^1 f(t)\,dt$
By differentiate both side, we have $f(x) = -f(x)$
Hence,$f(x) = 0$ for any $x \in [0,1]$

6. (a)

$$\int_0^3 \frac{2(x-1)}{(x^2+3)(x+1)^2}\,dx$$
$$= \int_0^3 \frac{1}{x^2+3} - \frac{1}{(x+1)^2}\,dx$$
$$= \left.\frac{1}{(x+1)}\right|_0^3 + \int_0^3 \frac{1}{x^2+3}\,dx \quad \text{Put} \quad x = \sqrt{3}\tan y$$
$$= \frac{1}{4} - 1 + \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{3}}\,dy \qquad\qquad dx = \sqrt{3}\sec^2 y\,dy$$
$$= -\frac{3}{4} + \frac{\pi}{3\sqrt{3}}$$

(b)

$$\int_0^{\frac{\pi}{2}} (\sin(x)+\cos(x))^2\,dx$$
$$= \int_0^{\frac{\pi}{2}} \sin^2 x + 2\sin x\cos x + \cos^2 x\,dx$$
$$= \int_0^{\frac{\pi}{2}} 1 + \sin 2x\,dx$$
$$= x - \left.\frac{\cos 2x}{2}\right|_0^{\frac{\pi}{2}}$$
$$= 1 + \frac{\pi}{2}$$

(c)

$$\int x\sin^2(x^2)\,dx \qquad \text{Put} \quad y = x^2$$
$$= \frac{1}{2}\int \sin^2(y)\,dy \qquad dy = 2x\,dx$$
$$= \frac{1}{2}\int \frac{1-\cos 2y}{2}\,dy$$
$$= \frac{y}{4} - \frac{\sin 2y}{8} + C$$
$$= \frac{x^2}{4} - \frac{\sin 2x^2}{8} + C$$

(d)

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \cos^6(x)\,dx \\
=&\int_0^{\frac{\pi}{2}} (\frac{1+\cos 2x}{2})^3\,dx \\
=&\int_0^{\frac{\pi}{2}} \frac{1+3\cos 2x+3\cos^2 2x+\cos^3 2x}{8}\,dx \\
=&\frac{x}{8}+\frac{3\sin 2x}{16}\Big|_0^{\frac{\pi}{2}}+\frac{3}{8}\int_0^{\frac{\pi}{2}} \frac{1+\cos 4x}{2}\,dx+\frac{1}{8}\int_0^{\frac{\pi}{2}} \cos^3 2x\,dx \\
=&\frac{5\pi}{32}
\end{aligned}$$

(e)

$$\begin{aligned}
&\int \frac{2dx}{\cot(\frac{x}{2})+\tan(\frac{x}{2})} \\
=&\int \frac{2}{\frac{\cos(\frac{x}{2})}{\sin(\frac{x}{2})}+\frac{\sin(\frac{x}{2})}{\cos(\frac{x}{2})}} \\
=&\int \frac{2\sin(\frac{x}{2})\cos(\frac{x}{2})}{\cos^2(\frac{x}{2})+\sin^2(\frac{x}{2})}\,dx \\
=&\int \sin x\,dx \\
=&-\cos x + C
\end{aligned}$$

(f) Consider $f(x)=\sin 2x-\sin x=(2\cos x-1)(\sin x)$

$f(x)\geqslant 0$ if $x\in[0,\frac{\pi}{3}]$

$f(x)\leqslant 0$ if $x\in[\frac{\pi}{3},\pi]$

$$\int_0^{\pi} |\sin 2x - \sin x|\, dx$$
$$= \int_0^{\frac{\pi}{3}} \sin 2x - \sin x\, dx + \int_{\frac{\pi}{3}}^{\pi} \sin x - \sin 2x\, dx$$
$$= \Big(-\frac{\cos 2x}{2} + \cos x \Big)\Big|_0^{\frac{\pi}{3}} + \Big(-\cos x + \frac{\cos 2x}{2} \Big)\Big|_{\frac{\pi}{3}}^{\pi}$$
$$= \frac{5}{2}$$

(g)

$$\int_0^{\pi} \Big| |\sin(2x)| - \sin(x) \Big|\, dx$$
$$=7 \int_0^{\frac{\pi}{3}} \sin 2x - \sin x\, dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x - \sin 2x\, dx + \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \sin 2x + \sin x\, dx$$
$$+ \int_{\frac{2\pi}{3}}^{\frac{\pi}{2}} (-\sin 2x - \sin x)\, dx$$
$$=1$$

(h)

$$\int \frac{1}{x} \ln(x)\, dx = (\ln x)^2 - \int \ln(x) \cdot \frac{1}{x}\, dx$$
$$\int \frac{1}{x} \ln(x)\, dx = \frac{1}{2} (\ln x)^2$$
$$\int \frac{1}{x^2} \ln(x)\, dx = -\frac{1}{x} \ln x + \int \frac{1}{x^2}\, dx$$
$$= -\frac{1}{x} \ln x - \frac{1}{x} + C$$
$$\int (\frac{1}{x} + \frac{1}{x^2}) \ln x\, dx = \frac{1}{2} (\ln x)^2 - \frac{1}{x} \ln x - \frac{1}{x} + C$$

(i)

$$\begin{aligned}
&\int_0^{\frac{\pi}{4}} \sec^4 x \, dx \qquad \text{Put} \quad y = \tan x \\
&= \int_0^1 y^2 + 1 \, dy \qquad\qquad dy = \sec^2 x \, dx \\
&= \frac{4}{3}
\end{aligned}$$

(j)

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \sqrt{\frac{1 - \sin x}{1 + \sin x}} \, dx \qquad\qquad \text{Put} \quad y = \sin x \\
&= \int_0^1 \sqrt{\frac{1 - y}{1 + y}} \cdot \frac{1}{\sqrt{1 - y^2}} \, dy \qquad\qquad dy = \cos x \, dx \\
&= \int_0^1 \frac{1}{1 + y} \, dy \\
&= \ln(2)
\end{aligned}$$

(k)

$$\begin{aligned}
&\int_0^{\frac{\pi}{6}} \sin x \tan x \, dx \\
&= \int_0^{\frac{\pi}{6}} \frac{\sin^2 x}{\cos x} \, dx \\
&= \int_0^{\frac{\pi}{6}} \frac{1 - \cos^2 x}{\cos x} \, dx \\
&= \int_0^{\frac{\pi}{6}} \sec x - \cos x \, dx \\
&= \ln|\sec x + \tan x| - \sin x \Big|_0^{\frac{\pi}{6}} \\
&= \ln(\sqrt{3}) - \frac{1}{2}
\end{aligned}$$

(l)

$$\begin{aligned}
&\int \frac{1+\cos x}{x+\sin x}\,dx && \text{Put} \quad y = x+\sin x \\
&= \int \frac{1}{y}\,dy && dy = 1+\cos x\,dx \\
&= \ln|y| + C \\
&= \ln|x+\sin x| + C
\end{aligned}$$

(m)

$$\begin{aligned}
&\int \frac{x^5}{x^3-1}\,dx \\
&= \int x^2 + \frac{x^2}{x^3-1}\,dx \\
&= \frac{x^3}{3} + \frac{1}{3}\ln|x^3-1| + C
\end{aligned}$$

7.

$$\begin{aligned}
\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx &= \int_{\pi}^{0} -\frac{(\pi-y)\sin(\pi-y)}{1+\cos^2(\pi-y)}\,dy \quad \text{Put} \quad x = \pi - y \\
&= \int_0^{\pi} \frac{(\pi-y)\sin y}{1+\cos^2 y}\,dy \qquad dx = -dy \\
\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx + \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx &= \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}\,dx \\
&= -\pi\tan^{-1}(\cos x)\Big|_0^{\pi} \\
&= \frac{\pi^2}{2} \\
\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx &= \frac{\pi^2}{4}
\end{aligned}$$

8. (a)

$$\begin{aligned}
&\int_{\frac{\pi}{2}}^{\pi} \frac{\sin^4 x}{\sin^4 x+\cos^4 x}\,dx && \text{Put} \quad x = \frac{\pi}{2} + y \\
&= \int_0^{\frac{\pi}{2}} \frac{\cos^4 y}{\sin^4 y+\cos^4 y}\,dy && dx = dy
\end{aligned}$$

(b)

$$\int_0^{\pi} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\, dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\, dx + \int_{\frac{\pi}{2}}^{\pi} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\, dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\, dx + \int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\sin^4 x + \cos^4 x}\, dx$$
$$= \frac{\pi}{2}$$

(c)

$$\int_0^{\pi} \frac{x \sin^4 x}{\sin^4 x + \cos^4 x}\, dx \qquad \text{Put} \quad x = \pi - y$$
$$= \int_{\pi}^{0} -\frac{(\pi - y)\sin^4 y}{\sin^4 y + \cos^4 y}\, dy \qquad dx = -dy$$
$$= \int_0^{\pi} \frac{(\pi - x)\sin^4 x}{\sin^4 x + \cos^4 x}\, dx$$
$$\int_0^{\pi} \frac{x \sin^4 x}{\sin^4 x + \cos^4 x}\, dx + \int_0^{\pi} \frac{(\pi - x)\sin^4 x}{\sin^4 x + \cos^4 x}\, dx$$
$$= \int_0^{\pi} \frac{\pi \sin^4 x}{\sin^4 x + \cos^4 x}\, dx$$
$$= \frac{\pi^2}{2}$$
$$\int_0^{\pi} \frac{x \sin^4 x}{\sin^4 x + \cos^4 x}\, dx = \frac{\pi^2}{4}$$

9. (a)

$$\sin x \cos x + \sin x \cos 3x + \ldots + \sin x \cos(2n-1)x$$
$$= \frac{1}{2}(\sin 2x - \sin 0) + \frac{1}{2}(\sin 4x - \sin 2x) + \ldots + \frac{1}{2}(\sin 2nx - \sin(2n-2)x)$$
$$= \frac{\sin 2nx}{2}$$
$$\cos x + \ldots + \cos(2n-1)x = \frac{\sin 2nx}{2 \sin x}$$

(b)

$$\begin{aligned}
&\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin 2nx}{\sin x}\,dx \\
=&\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2(\cos x + \ldots + \cos(2n-1)x)\,dx \\
=&2\cdot\left(\sin x + \frac{\sin 3x}{3} + \ldots + \frac{\sin(2n-1)x}{2n-1}\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}
\end{aligned}$$

10. (a) $F(x) = \int_a^x f(t)\,dt$ and $F'(x) = f(x) \geqslant 0$

Hence, $F(x)$ is increasing on $[a,b]$ and $F(a) \leqslant F(x) \leqslant F(b)$ for any $x \in [a,b]$

Since $F(a) = F(b) = 0$, we can conclude that $F(x) = 0$ for all $x \in [a,b]$

Therefore, $F'(x) = f(x) = 0$ for all $x \in [a,b]$

(b) Let $u(x) = g(x)$. Cosider $F(x) = \int_a^x g(t)^2\,dt$

F is an increasing function on $[a,b]$ and $\int_a^b g(t)^2\,dt = 0$

By part(a), we have $g(x)^2 = 0$ and hence $g(x) = 0$ for all $x \in [a,b]$

(c) i.

$$\begin{aligned}
&\int_a^b v(x)\,dx \\
=&\int_a^b h(x) - A\,dx \\
=&(b-a)A - (b-a)A \\
=&0
\end{aligned}$$

ii. Let$w(x) = h(x) - A$

$$\int_a^b h(x)(h9x) - a)\,dx = 0$$

$$\int_a^b (h(x) - A)^2 + A(h(x) - A)\,dx = 0$$

$$\int_a^b (h(x) - A)^2\,dx = \int_a^b A(A - h(x))\,dx$$
$$= 0$$

11. (a)

$$\frac{1 - t^{2n}}{1 - t^2} = \frac{(1 - t^2)(1 + t^2 + t^4 + \ldots + t^{2n-2})}{1 - t^2}$$
$$= 1 + t^2 + t^4 + \ldots + t^{2n-2}$$
$$\frac{1}{1 - t^2} = 1 + t^2 + t^4 + \ldots + t^{2n-2} + \frac{t^{2n}}{1 - t^2}$$

(b) i.

$$\int_0^x \frac{t}{1 - t^2}\,dt = -\frac{1}{2}\int_0^x \frac{-2t}{1 - t^2}\,dt$$
$$= -\frac{1}{2}\ln(1 - t^2) bigg|_0^x$$
$$= \ln\frac{1}{\sqrt{1 - x^2}}$$

ii.

$$\int_0^x \frac{t^{2n+1}}{1 - t^2}\,dt = \int_0^x \frac{t}{1 - t^2} - (t + t^3 + t^5 + \ldots + t^{2n-1})\,dt$$
$$= \ln\frac{1}{\sqrt{1 - x^2}} - (\frac{x^2}{2} + \frac{x^4}{4} + \ldots + \frac{x^{2n}}{2n})$$

(c) $\int_0^x 0\,dt \leqslant \int_0^x \frac{t^{2n+1}}{1-t^2}\,dt \leqslant \int_0^x \frac{t^{2n+1}}{1-x^2}\,dt$

$0 \leqslant \ln\frac{1}{\sqrt{1-x^2}} - (\frac{x^2}{2} + \frac{x^4}{4} + \ldots + \frac{x^{2n}}{2n}) \leqslant \frac{x^{2n+2}}{(1-x^2)(2n+2)}$

Put $x = \frac{\sqrt{8}}{3}$

we have $0 \leqslant \ln 3 - \sum_{k=1}^{n} \frac{1}{2k}(\frac{8}{9})^k \leqslant \frac{9}{2n+2}(\frac{8}{9})^{n+1}$

$\lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty} \frac{9}{2n+2}(\frac{8}{9})^{n+1} = 0$

By sandwich theorem, $\lim_{n\to\infty} \ln 3 - \sum_{k=1}^{n} \frac{1}{2k}(\frac{8}{9})^k = 0$

Hence $\sum_{k=1}^{\infty} \frac{1}{2k}(\frac{8}{9})^k = \ln 3$